

# PSEUDO-COMPLEX GENERAL RELATIVITY

Peter O. Hess<sup>1</sup> and Walter Greiner<sup>2</sup>

<sup>1</sup>*Instituto de Ciencias Nucleares, UNAM, Circuito Exterior, C.U., A.P. 70-543, 04510 México D.F., Mexico*

<sup>2</sup>*Frankfurt Institute of Advanced Studies, Johann Wolfgang Goethe Universität, Ruth-Moufang-Str. 1, 60438 Frankfurt am Main, Germany*

Received Day Month Year

Revised Day Month Year

An extension of the theory of General Relativity is proposed, based on pseudo-complex space-time coordinates. The new theory corresponds to the introduction of two, in general different, metrics which are connected through specific conditions. A pseudo-complex Schwarzschild solution is constructed, which does not suffer any more by a singularity. The solution indicates a minimal radius for a heavy mass object. Consequences for the redshift and possible signatures for its observation are discussed.

*Keywords:* General Relativity, pseudo-complex, extension of General Relativity

PACS numbers: 02.40.ky, 98.80.-k

## 1. Introduction

Several attempts have been made to generalize the Theory of General Relativity (GR) by extending algebraically the real coordinates of space-time to larger domains. For example, in Refs. <sup>1,2</sup> complex coordinates were proposed in an attempt to unify GR with electro-magnetism. Recently, in <sup>3</sup>, a more profound description with complex variables has been given. In Refs. <sup>4,5,6,7</sup> the coordinates were instead extended to so-called hypercomplex variables (an equivalent notation for pseudo-complex). There are many others, like <sup>8,9,10,11,12,13,14,15,16,17,18</sup>, which in addition exploit Born's principle of complementarity <sup>19,20</sup>. They have in common that a maximal acceleration appears in the theory, corresponding to a minimal length scale  $l$ .

In <sup>21</sup> it was shown that not all algebraic extensions of the coordinates make sense. Only real and pseudo-complex coordinates allow for physical solutions, i.e. the non-appearance of ghost solutions. This finding suggests to study in more detail the algebraic extension of the space-time coordinates to pseudo-complex variables, also investigating alternative proposals.

In <sup>22,23</sup> a pseudo-complex extension of Field Theory was presented, with the intention to investigate a modified dispersion relation, which allows a shift of the GZK limit <sup>24,25</sup> in the spectrum of high energy cosmic rays. Quite interesting, this theory is automatically regularized! Deviations of some cross sections, with respect

to standard results, were calculated resulting in a theoretically deduced upper limit of the minimal length scale  $l$  of approximately  $10^{-28}$  cm. The approach presented in <sup>22,23</sup>, partially relies on proposals made in <sup>26,27</sup>. The theory does conserve Lorentz invariance, which is attractive. Usually, the appearance of a minimal length is identified with the violation of Lorentz invariance. In contrast, in <sup>22,23</sup> the minimal length scale appears as a *scalar parameter* in the theory, not affected by Lorentz contraction.

The fact that pseudo-complex variables seem to be useful in distinct areas of physics, justifies again a deeper investigation. Our motivation is to study the algebraic extension of the space-time coordinates to pseudo-complex variables. The questions posed are: Is there a possibility to formulate this extension in a consistent manner? Because in field theory the appearance of a minimal length scale renders it regularized, hence: does the same happen in GR? Do singular solutions render non-singular? What is the role of the minimal length scale? What is the structure of the equivalent Schwarzschild solution, i.e., of a spherical symmetric large mass? Do black holes become gray?

We can not answer all these questions in this contribution and refer to a later publication. Nevertheless, some questions will be addressed: We will be able to formulate in a consistent way the extension of GR to pseudo-complex numbers. A solution for a spherical symmetric mass distribution, equivalent to the standard Schwarzschild solution, will be presented.

We proceed as follows: Based on the Refs. <sup>22,23</sup>, we will extend GR to a pseudo-complex description of the coordinates, using a different approach as in <sup>4,5,6,7</sup>. It will include a new variational principle, as proposed in <sup>26,27</sup>. One advantage of the new formulation is that it does not rely on the bundle frame language of differential geometry (DG), making it transparent for non-experts in DG. It will consist of twice the GR, defined in two separate spaces of the quasi phase space (its variables are coordinates and velocities, not momenta). Afterward, both spaces will be connected via the new variational principle.

As will be seen, a consistent formulation can be found and the results imply the elimination of singular solutions in GR, in particular there will be no Schwarzschild horizon. The deviation of the redshift in the new theory from that in standard GR will be determined. Whether our theory is realized in nature, has to be verified by experiment. It is, nevertheless, a valid possibility in the search of extensions of standard GR.

The paper is structured as follows: In section II a short review on pseudo-complex variables is given. In section III the extended formulation of GR is presented and in section IV the analogue to the standard Schwarzschild solution will be constructed, which is singularity free. (We will still refer to it as the Schwarzschild solution.) In addition, in a subsection, the consequences of the pseudo-complex Schwarzschild solution are discussed. In section V the redshift is calculated and observable deviations from standard GR are estimated. Finally, in section VI conclusions are drawn.

Most of the formulation is done in complete analogy to the book of Adler, Bazin and Schiffer<sup>28</sup>. We will simply refer to it at several places. In this way we avoid unnecessary repetition of the literature and gain, were it is convenient, didactical clarity.

## 2. Pseudo-Complex Variables

Here we give a brief resumé on pseudo-complex variables, helpful to understand the steps presented in this contribution. The formulas, presented here, can be used without going into the details. A more profound introduction to pseudo-complex variables is given in<sup>22,23</sup>, which can be consulted for better understanding.

The pseudo-complex variables are also known as *hyperbolic*<sup>4,5</sup>, *hypercomplex*<sup>29</sup> or *para-complex*<sup>30</sup>. We will continue to use the term pseudo-complex.

The pseudo-complex variables are *defined* via

$$X = x_1 + Ix_2 \quad , \quad (1)$$

with  $I^2 = 1$ . This is similar to the common complex notation except for the different behavior of  $I$ . An alternative presentation is to introduce the operators

$$\sigma_{\pm} = \frac{1}{2} (1 \pm I) \quad (2)$$

with

$$\sigma_{\pm}^2 = 1 \quad , \quad \sigma_+ \sigma_- = 0 \quad . \quad (3)$$

The  $\sigma_{\pm}$  form a so called *zero divisor basis*, with the zero divisor defined in mathematical terms by  $\mathbf{P}^0 = \mathbf{P}_+^0 \cup \mathbf{P}_-^0$ , with  $\mathbf{P}_{\pm}^0 = \{X = \lambda \sigma_{\pm} | \lambda \in \mathbf{R}\}$ .

This basis is used to rewrite the pseudo-complex variables as

$$X = X_+ \sigma_+ + X_- \sigma_- \quad , \quad (4)$$

with

$$X_{\pm} = x_1 \pm x_2 \quad . \quad (5)$$

The pseudo-complex conjugate of a pseudo-complex variable is

$$X^* = x_1 - Ix_2 = X_+ \sigma_- + X_- \sigma_+ \quad . \quad (6)$$

The *norm* square of a pseudo-complex variable is given by

$$|X|^2 = XX^* = x_1^2 - x_2^2 \quad . \quad (7)$$

4 Peter O. Hess and Walter Greiner

This allows for the appearance of a positive, negative and null norm. Variables with a zero norm are members of the zero-divisor, i.e., they are either proportional to  $\sigma_+$  or  $\sigma_-$ .

*It is very useful to do all calculations within the zero divisor basis,  $\sigma_\pm$ , because all manipulations can be realized independently in both sectors (because  $\sigma_+\sigma_- = 0$ ).*

In each zero divisor component, differentiation and multiplication can be manipulated in the same way as with normal variables. For example, we have <sup>23</sup>

$$F(X) = F(X_+)\sigma_+ + F(X_-)\sigma_- \quad (8)$$

and a product of two functions  $F(X)$  and  $G(X)$  satisfies

$$F(X)G(X) = F(X_+)G(X_+)\sigma_+ + F(X_-)G(X_-)\sigma_- \quad . \quad (9)$$

Differentiation is defined as

$$\frac{DF(X)}{DX} = \lim_{\Delta X \rightarrow 0} \frac{F(X + \Delta X) - F(X)}{\Delta X} \quad , \quad (10)$$

where  $\Delta$  refers from here on to the pseudo-complex difference. The  $D$  refers to the partial differentiation or infinitesimal difference.

Finally, we resume some properties of the quasi phase space of a pseudo-complex four dimensional space-time. It is mainly for completeness and can be skipped by the non-interested reader. There are a set of four coordinates  $X^\mu = X_+^\mu\sigma_+ + X_-^\mu\sigma_-$  and four velocities  $U^\mu = U_+^\mu\sigma_+ + U_-^\mu\sigma_-$ . Considering the motion of a mass point, the coordinates are often written as  $X_\pm^\mu = (x^\mu \pm lu^\mu)$  and the velocities as  $U_\pm^\mu = (u^\mu \pm la^\mu)$ , where  $x^\mu$ ,  $u^\mu = \frac{dx^\mu}{d\tau}$  ( $\tau$  as the eigen-time) and  $a^\mu = \frac{du^\mu}{d\tau}$  are called the standard coordinates, velocities and accelerations of the mass point, respectively, along the world line. There are two separated quasi phase spaces, due to the division in  $\sigma_\pm$  components. One is built by the pair  $(X_+^\mu, U_+^\mu)$  and the other one by  $(X_-^\mu, U_-^\mu)$ . All manipulations are done independently in each subspace of the quasi phase space. As is well known, canonical transformations are generated by the members of the algebra of a symplectic group. Having  $n$  coordinates the symplectic group is  $Sp(n, R)$ . Therefore, canonical transformations in the extended description of space-time exhibit a direct product structure of the type  $Sp_+(4, R) \otimes Sp_-(4, R)$ , where  $Sp_\pm(4, R)$  is the symplectic group of the four dimensional space.

### 3. Formulation of Pseudo-Complex General Relativity

In a first step, the pseudo-complex metric function is constructed, which is a pseudo-holomorphic function, i.e., it satisfies the pseudo-complex Riemann-Cauchy conditions <sup>23</sup>

$$\begin{aligned}\frac{Dg_{\mu\nu}^R}{DX_1^\lambda} &= \frac{Dg_{\mu\nu}^I}{DX_2^\lambda} \\ \frac{Dg_{\mu\nu}^R}{DX_2^\lambda} &= \frac{Dg_{\mu\nu}^I}{DX_1^\lambda} \quad ,\end{aligned}\tag{11}$$

where  $g_{\mu\nu}^R$  is the pseudo-real and  $g_{\mu\nu}^I$  the pseudo-imaginary component, with  $X_1^\lambda = x^\lambda$  being the pseudo-real part and  $X_2^\lambda$  the pseudo-imaginary part of the 4-coordinate  $X^\lambda = X_1^\lambda + IX_2^\lambda$ .

If we would assume that  $g_{\mu\nu}$  does only depend on the pseudo-real part  $X_1^\lambda = x^\lambda$  of the coordinate, it would lead us to a non-holomorphic function  $g_{\mu\nu}$  (e.g., the first equation in (11) would yield zero on the right hand side, while the left hand side is different from zero). The condition of  $g_{\mu\nu}$  being a pseudo-holomorphic function is, therefore, of importance.

Taking into account that differentiating a pseudo-complex function with respect to a variable follows the same rules as differentiating a normal function with respect to a variable, we can try to formulate GR following *the same steps* as indicated in 28. The important difference is that the metric is pseudo-complex. It is defined as

$$g_{\mu\nu} = g_{\mu\nu}^+ \sigma_+ + g_{\mu\nu}^- \sigma_- \quad .\tag{12}$$

It is a function of the pseudo-complex space-time variables  $X_\pm^\lambda$  and, thus, also a function of the coordinates and the velocities.

The differential length element squared is given by

$$\begin{aligned}d\omega^2 &= g_{\mu\nu}(X)DX^\mu DX^\nu \\ &= g_{\mu\nu}^+(X_+)DX_+^\mu DX_+^\nu \sigma_+ \\ &\quad + g_{\mu\nu}^-(X_-)DX_-^\mu DX_-^\nu \sigma_- \quad .\end{aligned}\tag{13}$$

The division is within the zero-divisor basis, similar to Eqs. (8) and (9). Interchanging in (13) the dummy indices  $\mu$  and  $\nu$  leads to

$$d\omega^2 = g_{\nu\mu}DX^\nu DX^\mu = g_{\nu\mu}DX^\mu DX^\nu \quad .\tag{14}$$

Comparing it to (13) requires that the metric is symmetric, i.e.,

$$g_{\mu\nu} = g_{\nu\mu} \quad .\tag{15}$$

*Because the  $\sigma_\pm$  parts are linearly independent, it implies that we can formulate a theory of General Relativity in each of the  $\sigma_+$  and the  $\sigma_-$  components.* Afterward, we have to connect both sectors, as will be explained further below. The advantage of the independent formulation lies in the fact that the GR in each sector will be analogue to the standard formulation.

For example, a parallel displacement of a pseudo-complex vector  $\xi^i$  is given by

$$\begin{aligned} D\xi^\mu &= \Gamma_{\nu\lambda}^\mu DX^\nu \xi^\lambda \\ &= \Gamma_{\nu\lambda}^{+\mu} DX_+^\nu \xi_+^\lambda \sigma_+ + \Gamma_{\nu\lambda}^{-\mu} DX_-^\nu \xi_-^\lambda \sigma_- \\ &= d\xi_+^\mu \sigma_+ + d\xi_-^\mu \sigma_- \quad , \end{aligned} \quad (16)$$

where  $DX^\nu$  refers to the change of the pseudo-complex coordinate  $X^\nu$  and  $\xi^\mu$  are the components of a vector, which is parallel displaced. The connections  $\Gamma_{\nu\lambda}^\mu$  are symmetric in their lower indices. The same arguments as in <sup>28</sup> leads to the now *pseudo-complex Christoffel symbols of the second kind*, starting from the condition that the pseudo-complex line-squared element  $d\omega^2$  is required to be invariant under the transformation of the coordinates. The pseudo-complex *Christoffel symbols of the second kind* are given by

$$\begin{aligned} \Gamma_{\mu\nu}^\lambda &= - \left\{ \begin{array}{cc} \lambda & \\ \nu & \mu \end{array} \right\} \\ &= - \left\{ \begin{array}{cc} \lambda & \\ \nu & \mu \end{array} \right\}_+ \sigma_+ - \left\{ \begin{array}{cc} \lambda & \\ \nu & \mu \end{array} \right\}_- \sigma_- \quad , \end{aligned} \quad (17)$$

which can be written in terms of *Christoffel symbols of the first kind* <sup>28</sup> as

$$\Gamma_{\mu\nu}^{\pm\lambda} = - \left\{ \begin{array}{cc} \lambda & \\ \nu & \mu \end{array} \right\}_\pm = - g^{\lambda\kappa} [\nu\mu, \kappa]_\pm \quad . \quad (18)$$

The *Christoffel symbol of the first kind* are defined as <sup>28</sup>

$$[\mu\nu, \kappa] = \frac{1}{2} \left( \frac{Dg_{\mu\kappa}}{DX^\nu} + \frac{Dg_{\nu\kappa}}{DX^\mu} - \frac{Dg_{\mu\nu}}{DX^\kappa} \right) \quad . \quad (19)$$

The expression  $\frac{Dg_{\mu\lambda}}{DX^\nu} = g_{\mu\lambda|\nu}$  denotes the pseudo-complex derivative of  $g_{\mu\lambda}$  with respect to  $X^\nu$ .

The 4-derivative of a contravariant vector is given by

$$\begin{aligned} \xi_{||\nu}^\mu &= \xi_{|\nu}^\mu + \left\{ \begin{array}{cc} \mu & \\ \nu & \lambda \end{array} \right\} \xi^\lambda \\ &= \left( \xi_{+|\nu}^\mu + \left\{ \begin{array}{cc} \mu & \\ \nu & \lambda \end{array} \right\}_+ \xi_+^\lambda \right) \sigma_+ \\ &\quad + \left( \xi_{-|\nu}^\mu + \left\{ \begin{array}{cc} \mu & \\ \nu & \lambda \end{array} \right\}_- \xi_-^\lambda \right) \sigma_- \quad , \end{aligned} \quad (20)$$

where  $\xi_{|\nu}^\mu = \frac{D\xi^\mu}{DX^\nu}$ . The rules for deriving covariant vectors and tensors can be directly copied from <sup>28</sup>.

An important point is that in this new formulation the 4-divergence of the metric will again be zero! To show this, we copy the arguments, as given in <sup>28</sup>, chapter 3. We have

$$g_{\mu\nu|\lambda} - g_{\mu\kappa} \left\{ \begin{matrix} \kappa \\ \nu \quad \lambda \end{matrix} \right\}_{\pm} = [\mu\lambda, \nu]_{\pm} \quad , \quad (21)$$

where the symmetry property of the metric tensor was used. Eq. (21) is proved by substituting the Christoffel symbol of the second kind (18) and using the definition of the Christoffel symbol of the first kind (19).

Using Eq. (21), the divergence of  $g_{\mu\nu}^{\pm}$  can be rewritten as

$$\begin{aligned} g_{\mu\nu||\lambda}^{\pm} &= g_{\mu\nu|\lambda}^{\pm} - \left\{ \begin{matrix} \kappa \\ \nu \quad \lambda \end{matrix} \right\}_{\pm} g_{\mu\kappa}^{\pm} - \left\{ \begin{matrix} \kappa \\ \mu \quad \lambda \end{matrix} \right\}_{\pm} g_{\kappa\nu}^{\pm} \\ &= [\mu\lambda, \nu]_{\pm} - g_{\kappa\nu}^{\pm} \left\{ \begin{matrix} \kappa \\ \mu \quad \lambda \end{matrix} \right\}_{\pm} \end{aligned} \quad (22)$$

Utilizing the definition of the Christoffel symbol of the second kind (see above), this expression is identical to zero. Thus, also the 4-divergence of the pseudo-complex metric is zero:

$$g_{\mu\nu||\lambda} = g_{\mu\nu||\lambda}^{+} \sigma_{+} + g_{\mu\nu||\lambda}^{-} \sigma_{-} = 0 \quad , \quad (23)$$

or equivalently

$$g_{\mu\nu||\lambda}^{\pm} = 0 \quad , \quad (24)$$

where the derivative is now with respect to the coordinates  $X_{\pm}^{\lambda}$ .

*This result is very important:* It is a necessary requirement for the principle of General Relativity. (The tensor  $g_{\mu\nu}$  is invariant under the action of the four-derivative, thus it is invariant under a parallel displacement. The same holds trivially for the operator  $I$ , because it is constant. Thus, we have an almost product structure <sup>31</sup>.)

This also leads to two different kinds of four derivatives, one for the  $\sigma_{+}$  and the other one for the  $\sigma_{-}$  component.

In order to proceed further, we need to introduce an important difference to the treatment of standard GR. It is the *change in the variational principle*: Up to now, it seems that we have only a double, parallel, formulation of GR, one in the  $\sigma_{+}$  and the other one in the  $\sigma_{-}$  component. In the next step we show how both zero-divisor components are *linked together*. We will follow a suggestion given in <sup>22,23,26,27</sup>:

Following the Lagrange formulation and denoting by  $L$  the Lagrangian within an integral, we have the action

8 *Peter O. Hess and Walter Greiner*

$$S = \int L d\tau \quad . \quad (25)$$

The *variational procedure is now modified to*<sup>23</sup>

$$\delta S = \delta \int L d\tau \in \mathbf{P}^0 \quad , \quad (26)$$

with  $\mathbf{P}^0$  being the zero divisor (numbers linear in  $\sigma_+$  or  $\sigma_-$  only<sup>23</sup>). One argument is that the zero divisor branch consists of numbers which have a zero norm and in this sense it represents a generalized zero.

To illustrate it more, suppose we would require that the variation of the action is exactly zero, then one gets that  $\delta S = \delta S_+ \sigma_+ + \delta S_- \sigma_- = 0$ , or  $\delta S_{\pm} = 0$ . In other words, one would obtain simply a double formulation of GR. However, if it is required that the variation of the action is within the zero-divisor branch, then both components are linked and only then it makes sense to obtain a new, modified theory of General Relativity.

The variation of the action leads to

$$\frac{D}{Ds} \left( \frac{DL}{DX^\mu} \right) - \frac{DL}{DX^\mu} \in \mathbf{P}^0 \quad , \quad (27)$$

with  $s$  as some curve parameter, which can be the eigen-time  $\tau$ . Note, that the right hand side has to be in the zero divisor, i.e., it is proportional either to  $\xi_\mu \sigma_-$  or  $\xi_\mu \sigma_+$ , with  $\xi_\mu$  a real or normal complex number or function. These  $\xi$ 's can be used as an additional freedom to fix solutions of the equations of motion and *will play a crucial role*.

As a Lagrangian one can use the length element, which leads to the equation of geodesics (in fact two, for each component in the zero-divisor basis):

$$\ddot{X}^\mu + \left\{ \begin{matrix} \mu \\ \nu \lambda \end{matrix} \right\} \dot{X}^\nu \dot{X}^\lambda \in \mathbf{P}^0 \quad . \quad (28)$$

This, however, assumes a test-particle description, as explained in Ref.<sup>28</sup>. Expressing  $L$  in terms of a curvature tensor, which is independent to the use of a test particle, we obtain<sup>28</sup> for a matter free space

$$G_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} \in \mathbf{P}^0 \quad , \quad (29)$$

modified by the zero divisor on the right hand side.  $G_{\mu\nu}$  is the pseudo-complex Einstein tensor,  $\mathcal{R}_{\mu\nu}$  is the Ricci tensor, defined in the same way as in standard GR, with the difference that now it is pseudo-complex. The  $\mathcal{R}$  is the Riemann curvature (also known as *scalar curvature*). The Lagrangian used has the form  $L$



$= \sqrt{-g}\mathcal{R}$ , with  $g$  being the determinant of the metric tensor. In standard GR the equations of motion reduce, for a matter free space, to the Ricci tensor equal to zero, i.e.  $\mathcal{R}_{\mu\nu} = 0$ . In our procedure, this is extended to

$$\mathcal{R}_{\mu\nu} = \mathcal{R}_{\mu\nu}^+ \sigma_+ + \mathcal{R}_{\mu\nu}^- \sigma_- \in \mathcal{P}^0 \quad . \quad (30)$$

Comparing it with (29) leads to

$$\mathcal{R} = \mathcal{R}_+ \sigma_+ + \mathcal{R}_- \sigma_- = 0 \quad . \quad (31)$$

*This is an important result. It means that the space has still a local zero scalar curvature.* This gives us an additional and necessary relation which will fix the functions appearing on the right hand side of the equation of motion.

Some words have to be said about the integrability of the system: The assumption that  $\mathcal{R}_{\mu\nu}$  is not exactly zero but in the zero divisor basis (e.g., proportional to  $\sigma_-$ ) implies that a pseudo-complex vector parallel displaced along two curves leads to, in general, two different vectors. However, the two vectors differ only by a component within the zero divisor basis. If  $\xi_1^\mu$  denotes the vector obtained after parallel displacement along a curve  $C_1$  and  $\xi_2^\mu$  is the vector obtained after parallel displacement along the curve  $C_2$ , then  $\xi_1^\mu - \xi_2^\mu \in \mathcal{P}^0$ . The difference vector has zero norm. *In the pseudo-complex space we, therefore, consider all vectors to be equivalent which differ only by a component in the zero divisor basis, where by convention we will choose it to be proportional to  $\sigma_-$ .* This proposal is a generalized definition of integrability. As we will see in section IV, it leads finally to a standard description of GR, were integrability is assured, while the effect of the pseudo-complex description is surviving through a particular contribution in the metric components. Thus, the description presented here, shows a possible alternative description of GR by making a detour to pseudo-complex numbers, an idea not thought before.

*As just defined, in the equation of motion we will assume for convenience*<sup>22,23</sup>, *that the right hand side is proportional to  $\sigma_-$ .* The assumption of being proportional to  $\sigma_+$  leads to a symmetric description.

### 3.1. Further properties of the metric

In this section we discuss further properties related to the metric, useful for subsequent calculations.

In the zero-divisor basis, the metric is given by Eq. (12). The product with its inverse,  $g^{\mu\nu}$ , yields

$$\begin{aligned} g^{\mu\nu} g_{\nu\lambda} &= g_+^{\mu\nu} g_{\nu\lambda}^+ \sigma_+ + g_-^{\mu\nu} g_{\nu\lambda}^- \sigma_- \\ &= \delta_{\mu\lambda} (\sigma_+ + \sigma_-) = \delta_{\mu\lambda} \quad . \end{aligned} \quad (32)$$

This choice contains two, in general, different metrics in the  $\sigma_\pm$  parts.

The pseudo-real and pseudo-imaginary part of the metric is related to  $g_{\mu\nu}^+$  and  $g_{\mu\nu}^-$  by

$$\begin{aligned} g_{\mu\nu}^R &= \frac{1}{2} (g_{\mu\nu}^+ + g_{\mu\nu}^-) = g_{\mu\nu}^0 \\ g_{\mu\nu}^I &= \frac{1}{2} (g_{\mu\nu}^+ - g_{\mu\nu}^-) = h_{\mu\nu} \quad , \end{aligned} \quad (33)$$

where we introduce a new, more convenient notation in terms of an *average* metric  $g_{\mu\nu}^0$  and a *difference* metric  $h_{\mu\nu}$ . This leads to

$$g_{\mu\nu}^\pm = g_{\mu\nu}^0 \pm h_{\mu\nu} \quad . \quad (34)$$

The metric  $g_{\mu\nu}^\pm$  lowers the index of  $X_\pm^\mu$  and  $P_\pm^\mu$ , while  $g_{\pm}^{\mu\nu}$  raises the ones of  $X_\mu^\pm$  and  $P_\mu^\pm$ , i.e.,

$$\begin{aligned} X_\mu^\pm &= g_{\mu\nu}^\pm X_\pm^\nu \\ X_\pm^\mu &= g_{\pm}^{\mu\nu} X_\nu^\pm \end{aligned} \quad (35)$$

or equivalently

$$\begin{aligned} x_\mu \pm l u_\mu &= g_{\mu\nu}^\pm (x^\nu \pm l u^\nu) \\ x^\mu \pm l u^\mu &= g_{\pm}^{\mu\nu} (x_\nu \pm l u_\nu) \end{aligned} \quad (36)$$

and similar for  $P_\pm^\mu = p_\mu \pm l f_\mu$  and  $P_\mu^\pm = p^\mu \pm l f^\mu$ , with  $p^\mu$  as the linear momentum and  $f^\mu$  being an object with the units of a force. Note, that one has to apply  $g_{\mu\nu}^\pm$  on  $X_\pm^\mu$  and not separately on  $x^\mu$  and  $u^\mu$ .

From the former equations (after subtracting and adding the first two equations in (36), in order to solve for  $x_\mu$  and  $p_\mu$ ) we obtain

$$\begin{aligned} x_\mu &= \frac{1}{2} (g_{\mu\nu}^+ + g_{\mu\nu}^-) x^\nu + l \frac{1}{2} (g_{\mu\nu}^+ - g_{\mu\nu}^-) u^\nu \\ &= g_{\mu\nu}^0 x^\nu + l h_{\mu\nu} u^\nu \\ l u_\mu &= \frac{1}{2} (g_{\mu\nu}^+ - g_{\mu\nu}^-) x^\nu + l \frac{1}{2} (g_{\mu\nu}^+ + g_{\mu\nu}^-) u^\nu \\ &= l g_{\mu\nu}^0 u^\nu + h_{\mu\nu} x^\nu \quad . \end{aligned} \quad (37)$$

One feature is that the raising and lowering of the indices can be applied only via a metric in the zero divisor components of  $X^\mu$  ( $X_\mu$ ), i.e., the *individual expressions of the coordinate* ( $x^\mu$ ,  $x_\mu$ ) *and velocities* ( $u^\mu$ ,  $u_\mu$ ) *are not contra- and covariant vectors any more*. The consequences of this have still to be explored. The exception happens in the limit when  $l$  and  $h_{\mu\nu}$  are zero, then  $g_{\mu\nu}^0$  ( $g_0^{\mu\nu}$ ) lower (raise) the components of the space-time and 4-velocity components.

The invariant generalized length element is given by Eq. (13). *Because it is an observable, it is required to be pseudo-real..* Thus,

$$d\omega^{*2} = d\omega^2 \quad . \quad (38)$$

From this condition we obtain the following relation

$$\begin{aligned} g_{\mu\nu}^+(X_+)DX_+^\mu DX_+^\nu \sigma_+ + g_{\mu\nu}^-(X_-)DX_-^\mu DX_-^\nu \sigma_- \\ = \\ g_{\mu\nu}^+(X_+)DX_+^\mu DX_+^\nu \sigma_- + g_{\mu\nu}^-(X_-)DX_-^\mu DX_-^\nu \sigma_+ \quad , \end{aligned} \quad (39)$$

or

$$g_{\mu\nu}^+(X_+)DX_+^\mu DX_+^\nu = g_{\mu\nu}^-(X_-)DX_-^\mu DX_-^\nu \quad . \quad (40)$$

Expressing the  $X_\pm^\mu$  in terms of  $x^\mu$  and  $u^\mu$  leads to

$$\begin{aligned} h_{\mu\nu} (dx^\mu dx^\nu + l^2 du^\mu du^\nu) \\ + l g_{\mu\nu}^0 (dx^\mu du^\nu + du^\mu dx^\nu) = 0 \quad . \end{aligned} \quad (41)$$

This is a generalized version for the "orthogonality" of  $dx$  and  $du$ . For the special case of a flat space ( $h_{\mu\nu} = 0$  and  $g_{\mu\nu}^0 = \eta_{\mu\nu}$ ) we arrive at the standard relation of  $dx_\mu du^\mu = 0$ . (We use the signature  $\eta_{\mu\nu} = (+, -, -, -)$ .)

With Eq. (41), the  $d\omega^2$  acquires the form

$$\begin{aligned} d\omega^2 = g_{\mu\nu}^0 (dx^\mu dx^\nu + l^2 du^\mu du^\nu) \\ + l h_{\mu\nu} (dx^\mu du^\nu + du^\mu dx^\nu) \quad . \end{aligned} \quad (42)$$

The new invariant length element can be simplified if only terms up to the order in  $l^0$  are considered. It leads to

$$d\omega^2 \approx g_{\mu\nu}^0 dx^\mu dx^\nu \quad . \quad (43)$$

As one can see, applying the above approximations *reduces the generalized length element  $d\omega^2$  to the standard form known for  $ds^2$* . This can be understood, considering that  $d\omega^2$  is related to  $ds^2$  by a factor of the type  $(1 - l^2 a^2)$ , with  $l$  as the minimal length parameter and  $a$  for the acceleration<sup>8,9,10,11,12,13,14,15</sup>. Taking into account only terms proportional to  $l^0$  reduces this factor to 1. The important point is that  $g_{\mu\nu}^0$  now depends on the differences between  $g_{\mu\nu}^\pm(X_\pm)$ , which contains remnant contributions from the pseudo-complex description. Note, that within this approximation we will return to the standard description of GR. What we will not discuss here are the contributions generated by corrections due to the minimal length element. We refer to a later publication.

This expression of the line element will play a crucial role in the next section, devoted to the discussion of the Schwarzschild solution within the pseudo-symplectic formulation.

#### 4. Pseudo-complex Schwarzschild solution

Again, we follow closely the book <sup>28</sup>, chapter 6.

In a first step, we deduce the pseudo-complex Christoffel symbols of the second kind, using the method described in <sup>28</sup>.

The length element is deduced by requiring that it is invariant under  $DX^0 \rightarrow -DX^0$ ,  $D\theta \rightarrow -D\theta$  and  $D\phi \rightarrow -D\phi$ . Following the same steps as in chapter 6 of <sup>28</sup>, the resulting pseudo-complex length element is given by

$$d\omega^2 = A(DX_0)^2 - B(DR)^2 - R^2 \left( (D\theta)^2 + \sin^2\theta (D\phi)^2 \right) . \quad (44)$$

The  $A_{\pm}$  and  $B_{\pm}$  functions are positive definite and one can write them as  $A_{\pm} = e^{\nu_{\pm}(R_{\pm})}$  and  $B_{\pm} = e^{\lambda_{\pm}(R_{\pm})}$ . Because of this, we can also write  $A = A_+\sigma_+ + A_-\sigma_-$  and  $B = B_+\sigma_+ + B_-\sigma_-$  as  $A = e^{\nu(R)}$  and  $B = e^{\lambda(R)}$  respectively. The same result is obtained when one parts from the line element in each zero divisor component ( $d\omega_{\pm}^2$ ) and imposes the same symmetry conditions to the corresponding variables. It is nothing but proceeding in each zero divisor component as in standard GR.

In order to deduce the Christoffel symbols, we recur to a trick by first determining the equation of geodesics. We use as a variational principle

$$\delta \int \left[ A\dot{X}_0^2 - B\dot{R}^2 - R^2 \left( \dot{\theta}^2 + \sin^2\theta \dot{\phi}^2 \right) \right] dp \in \mathcal{P}^0 , \quad (45)$$

where a dot above a variables indicates its derivative with respect to  $p$ , the curve parameter along the world line of particle (for example,  $p$  can be the eigen-time  $\tau$ , the arc length  $s$ , etc.). The variation leads to equations of motion of the form of (28). By our convention we will require that the  $\sigma_-$  component is different from zero. The right hand side of (28) can be a function in the pseudo-complex variables.

These equations have to be compared with those obtained from (45), which are

$$\begin{aligned} \ddot{X}^0 + \nu' \dot{R} \dot{X}^0 &= \xi^0 \sigma_- \\ \ddot{R} + \frac{1}{2} \lambda' \dot{R}^2 + \frac{1}{2} \nu' e^{\nu-\lambda} (\dot{X}^0)^2 - e^{-\lambda} R \dot{\theta}^2 \\ &\quad - R \sin^2\theta \dot{\phi}^2 e^{-\lambda} = \xi_R \sigma_- \\ \ddot{\theta} + \frac{2}{R} \dot{\theta} \dot{R} - \sin\theta \cos\theta \dot{\phi}^2 &= \xi_{\theta} \sigma_- \\ \ddot{\phi} + 2 \cot\theta \dot{\phi} \dot{\theta} + \frac{2}{R} \dot{R} \dot{\phi} &= \xi_{\phi} \sigma_- , \end{aligned} \quad (46)$$

for  $X^0$ ,  $R$ ,  $\theta$  and  $\phi$  respectively. A prime indicates the derivative with respect to  $R$ , e.g.,  $\nu' = \frac{D\nu}{DR}$ . On the right hand side of each equation we put an element of the

zero-divisor, such that it is proportional to  $\sigma_-$ , following our convention. The  $\sigma_{\pm}$  components of the  $\lambda$  and  $\nu$  are functions in the variables  $X_{\pm}^{\mu}$ . Comparing (46) with (28) gives us the Christoffel symbols of the second kind. The non-zero Christoffel symbols are given by

$$\begin{aligned}
\begin{Bmatrix} 0 \\ 1 \quad 0 \end{Bmatrix} &= \frac{1}{2}\nu' = \begin{Bmatrix} 0 \\ 0 \quad 1 \end{Bmatrix} \\
\begin{Bmatrix} 1 \\ 0 \quad 0 \end{Bmatrix} &= \frac{1}{2}\nu'e^{\nu-\lambda} \\
\begin{Bmatrix} 1 \\ 1 \quad 1 \end{Bmatrix} &= \frac{1}{2}\lambda' \\
\begin{Bmatrix} 1 \\ 2 \quad 2 \end{Bmatrix} &= -Re^{-\lambda} \\
\begin{Bmatrix} 1 \\ 3 \quad 3 \end{Bmatrix} &= -R\sin^2\theta e^{-\lambda} \\
\begin{Bmatrix} 2 \\ 2 \quad 1 \end{Bmatrix} &= \frac{1}{R} = \begin{Bmatrix} 2 \\ 1 \quad 2 \end{Bmatrix} \\
\begin{Bmatrix} 2 \\ 3 \quad 3 \end{Bmatrix} &= -\sin\theta \cos\theta \\
\begin{Bmatrix} 3 \\ 2 \quad 3 \end{Bmatrix} &= \cot\theta = \begin{Bmatrix} 3 \\ 3 \quad 2 \end{Bmatrix} \\
\begin{Bmatrix} 3 \\ 1 \quad 3 \end{Bmatrix} &= \frac{1}{R} = \begin{Bmatrix} 3 \\ 3 \quad 1 \end{Bmatrix} .
\end{aligned} \tag{47}$$

In the next step we use the proposed equation of motion, as given in Eq. (30) above, with the subsidiary condition for the curvature ( $\mathcal{R} = 0$ , see Eq. (31)). For that, we remind on the structure of the metric, which is

$$\begin{pmatrix} e^{\nu(R)} & 0 & 0 & 0 \\ 0 & -e^{\lambda(R)} & 0 & 0 \\ 0 & 0 & -R^2 & 0 \\ 0 & 0 & 0 & -R^2\sin^2\theta \end{pmatrix} \tag{48}$$

and its determinant is

$$g = -e^{\nu+\lambda}R^4\sin^2\theta . \tag{49}$$

For its logarithm we get

$$\ln\sqrt{-g} = \frac{\nu+\lambda}{2} + 2\ln R + \ln|\sin\theta| . \tag{50}$$

14 *Peter O. Hess and Walter Greiner*

The Ricci tensor is of the form

$$\begin{aligned}\mathcal{R}_{\mu\nu} = & \left\{ \begin{matrix} \beta \\ \beta & \nu \end{matrix} \right\}_{|\mu} - \left\{ \begin{matrix} \beta \\ \mu & \nu \end{matrix} \right\}_{|\beta} \\ & + \left\{ \begin{matrix} \beta \\ \tau & \mu \end{matrix} \right\} \left\{ \begin{matrix} \tau \\ \beta & \nu \end{matrix} \right\} \\ & - \left\{ \begin{matrix} \beta \\ \tau & \beta \end{matrix} \right\} \left\{ \begin{matrix} \tau \\ \mu & \nu \end{matrix} \right\} .\end{aligned}\quad (51)$$

Using (47) and the explicit expressions for the Christoffel symbols of Eq. (47), we obtain for the  $\mathcal{R}_{\mu\mu}$  components

$$\begin{aligned}\mathcal{R}_{00} &= -\frac{e^{\nu-\lambda}}{2} \left( \nu'' + \frac{\nu'^2}{2} - \frac{\lambda'\nu'}{2} + \frac{2\nu'}{R} \right) \\ \mathcal{R}_{11} &= \frac{1}{2} \left( \nu'' + \frac{\nu'^2}{2} - \frac{\lambda'\nu'}{2} - \frac{2\lambda'}{R} \right) \\ \mathcal{R}_{22} &= (e^{-\lambda}R)' - 1 \\ \mathcal{R}_{33} &= \sin^2\theta \left[ (e^{-\lambda}R)' - 1 \right] .\end{aligned}\quad (52)$$

With that, the equations for  $\mathcal{R}_{00}$  and  $\mathcal{R}_{11}$  are resulting:

$$\begin{aligned}\nu'' + \frac{1}{2}\nu'^2 - \frac{1}{2}\lambda'\nu' + \frac{2\nu'}{R} &= \xi_0\sigma_- \\ \nu'' + \frac{1}{2}\nu'^2 - \frac{1}{2}\lambda'\nu' - \frac{2\lambda'}{R} &= \xi_1\sigma_- .\end{aligned}\quad (53)$$

On the left hand side only functions in  $R$  appear. Hence, the  $\xi_0$  and  $\xi_1$  functions depend on  $R$  only, too. (In fact, because  $f(R)\sigma_- = f(R_-)\sigma_-$ , these functions depend only on  $R_-$ ).

Subtracting both equations and utilizing  $R\sigma_- = R_-\sigma_-$  yields:

$$\nu' + \lambda' = \frac{1}{2}R_- (\xi_0(R_-) - \xi_1(R_-)) \sigma_- .\quad (54)$$

The length element is a function in  $\lambda$  and  $\nu$ , i.e.,

$$\begin{aligned}d\omega^2 &= e^\nu (DX^0)^2 - e^\lambda (DR)^2 \\ &\quad - R^2 \left( (D\theta)^2 + \sin^2\theta (D\phi)^2 \right) .\end{aligned}\quad (55)$$

The solution (54) implies that

$$\begin{aligned}e^{-\lambda} &= e^{\nu - \int \frac{R(\xi_0 - \xi_1)}{2} dR_-} \sigma_- \\ &= e^{\nu_+} \sigma_+ + e^{\nu_- - \int \frac{R_-(\xi_0 - \xi_1)}{2} dR_-} \sigma_- .\end{aligned}\quad (56)$$

This gives us a restriction on  $\xi_0$  and  $\xi_1$ : Suppose, we get a sensible result for  $\lambda$ , which goes to zero for large values of  $r$ , such that  $e^\lambda \rightarrow 1$  and  $e^\nu \rightarrow 1$ . For that, we note that  $R_\pm = r \pm l\dot{r}$  ( $\dot{r} = \frac{dr}{d\tau}$  is the radial velocity), i.e., its pseudo-real component is  $r$ , while its pseudo-imaginary component is  $l\dot{r}$ , and  $R_+$  ( $R_-$ ) are the sum (difference) of these components. Then Eq. (56) implies that  $e^{\nu-}$  tends for large  $r$  to a value different from one. The limit 1 is only achieved if the integrand is set to zero, i.e.,

$$\xi_0 = \xi_1 \quad , \quad (57)$$

i.e.,  $\nu' = -\lambda'$ .

Substituting this into the second equation of (53), which originates from  $\mathcal{R}_{11}$ , we obtain

$$\begin{aligned} \lambda'' - \lambda'^2 + \frac{2\lambda'}{R} &= -\xi_1 \sigma_- \\ &= -\frac{e^\lambda}{R} (Re^{-\lambda})'' \quad . \end{aligned} \quad (58)$$

The last expression leads to

$$(Re^{-\lambda})' = \text{const} + \int R_- e^{-\lambda-} \xi_1(R_-) dR_- \sigma_- \quad . \quad (59)$$

Using the expression for  $\mathcal{R}_{22}$  yields

$$\begin{aligned} (e^{-\lambda} R)' &= (e^{-\lambda+} R_+)' \sigma_+ + (e^{-\lambda-} R_-)' \sigma_- \\ &= 1 + \xi_2 \sigma_- \quad , \end{aligned} \quad (60)$$

which determines the right hand side of (59). The main conclusion is that the expression on the left hand side of in Eq. (60) is proportional to 1 plus a function in  $R_-$  (i.e.,  $\xi_2(R_-)$  times  $\sigma_-$ , being an element of the zero divisor branch).

Integrating Eq. (60) yields

$$e^{-\lambda} R = R - 2\mathcal{M} + \int \xi_2(R_-) dR_- \sigma_- \quad , \quad (61)$$

where  $-2\mathcal{M} = -2(M_+ \sigma_+ + M_- \sigma_-)$  is a pseudo-complex integration constant.

For the  $\sigma_+$  part ( $\lambda \rightarrow \lambda_+$  and  $R \rightarrow R_+$ ), the equation is equivalent to the one in the book of Adler et al., i.e.,

$$e^{-\lambda_+} = 1 - \frac{2M_+}{R_+} \quad . \quad (62)$$

Let us, therefore, restrict to the  $\sigma_-$  part only. The  $\sigma_-$  part reads

$$e^{-\lambda_-} = 1 - \frac{2M_-}{R_-} + \frac{1}{R_-} \int \xi_2(R_-) dR_- \quad . \quad (63)$$

We have to substitute this expression into equation (58), which resulted from  $\mathcal{R}_{11}$ , i.e.,

$$\begin{aligned} \lambda_-'' - \lambda_-'^2 + \frac{2\lambda_-'}{R_-} &= -\frac{e^{\lambda_-}}{R_-} (R_- e^{-\lambda_-})'' \\ &= -\xi_1 \quad . \end{aligned} \quad (64)$$

This leads to

$$(R_- e^{-\lambda_-})'' = \xi_1(R_-) R_- e^{-\lambda_-(R_-)} \quad . \quad (65)$$

Utilizing Eq. (60), we arrive at

$$(1 + \xi_2(R_-))' = \xi_2'(R_-) = R_- \xi_1(R_-) e^{-\lambda_-(R_-)} \quad . \quad (66)$$

Using Eq. (63) on the right hand side, we get

$$\xi_2' = \xi_1 \left[ R_- - 2M_- + \int \xi_2 dR_- \right] \quad . \quad (67)$$

Note, that the dimension of  $\xi_0 = \xi_1$  is one over length squared (see Eq. (64)). In contrast  $\xi_2$  has no dimension (see Eq. (60)).

It remains to see what the other relations  $\mathcal{R}_{\mu\nu} \in \mathcal{P}^0$ , for  $(\mu, \nu) = (3, 3)$  and also for  $\mu \neq \nu$  give, considering only the  $\sigma_-$  part.

Following again the book of Adler et al., for  $(\mu, \nu) = (3, 3)$ , using Eq. (52), we get

$$\sin^2 \theta \left[ (R_- e^{-\lambda_-})' - 1 \right] = \xi_3 \quad , \quad (68)$$

where according to Eq. (60) the parenthesis [...] on the left side is just  $\xi_2$ . Therefore,  $\xi_3$  is of the form

$$\xi_3 = \xi_2 \sin^2 \theta \quad . \quad (69)$$

Thus we arrive at a set of relations for the  $\xi_k$  ( $k = 0, 1, 2, 3$ ). All can be expressed in terms of, e.g.,  $\xi_1$ .

As in Ref. <sup>28</sup>, all other components of  $R_{\mu\nu}$ , with  $\mu \neq \nu$  are identically zero, which is proved using the explicit expressions (51) of the Ricci tensor and the list of the non-zero components (47) of the Christoffel symbols of the second kind..

The  $\sigma_-$  component of  $e^{-\lambda}$  now reads (see (64))



$$\begin{aligned}
e^{-\lambda_-} &= 1 - \frac{2M_-}{R_-} + \frac{1}{R_-} \int \xi_2 dR_- \\
&= 1 - \frac{2M_-}{R_-} + \frac{\Omega}{R_-} \quad , \quad (70)
\end{aligned}$$

where the  $\Omega(R_-)$  function is defined as

$$\Omega = \int \xi_2 dR_- \quad . \quad (71)$$

The next and last step consists in applying the condition  $\mathcal{R} = 0$ , which relates the  $\xi_1$  with the  $\xi_2$  function. Using the metric (48) and that  $\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu}$ , being a scalar in the  $\sigma_{\pm}$  components, we obtain

$$\begin{aligned}
\mathcal{R} &= e^{-\nu}\mathcal{R}_{00} - e^{-\lambda}\mathcal{R}_{11} - \frac{1}{R^2}\mathcal{R}_{22} - \frac{1}{R^2\sin^2\theta}\mathcal{R}_{33} \\
&= \left(-e^{-\lambda}\xi_1 - \frac{2}{R^2}\xi_2\right)\sigma_- = \left(-e^{-\lambda_-}\xi_1 - \frac{2}{R_-^2}\xi_2\right)\sigma_- \\
&= 0 \quad , \quad (72)
\end{aligned}$$

where we have used on one side the relations between the  $\xi$ -functions and the components  $\mathcal{R}_{\mu\nu}$  of the Ricci tensor (52), (53), (60), (68) and on the other side the relations between the  $\xi_{\mu}$  functions (57), (69). The dependence of the  $\mathcal{R}_{\mu\mu}$  components on the  $\xi$  functions, which can be deduced from the equations in (52), are

$$\begin{aligned}
\mathcal{R}_{00} &= -\frac{1}{2}e^{\nu-\lambda}\xi_0\sigma_- \quad , \quad \mathcal{R}_{11} = \frac{1}{2}\xi_1\sigma_- \\
\mathcal{R}_{22} &= \xi_2\sigma_- \quad , \quad \mathcal{R}_{33} = \xi_3\sigma_- \quad . \quad (73)
\end{aligned}$$

We obtain from (72)

$$\xi_1 = -\frac{2e^{\lambda_-}}{R_-^2}\xi_2 \quad . \quad (74)$$

The result is substituted into (67), yielding

$$\xi_2' = -\frac{2}{R_-}\xi_2 \quad . \quad (75)$$

which is a differential equation for  $\xi_2$ , with the solution

$$\xi_2 = \frac{-B}{R_-^2} \quad , \quad (76)$$

where  $B$  is an integration constant. The minus sign is for convenience and can be understood further below. For the function  $\Omega$  (see (71)), Eq. (76) implies that

$$\Omega = \frac{B}{R_-} \quad . \quad (77)$$

In order to find restrictions for the integration constant  $B$ , we have to set some conditions, namely: i) Within the Schwarzschild radius, the expression (70) has to be positive, such that the time is defined in the usual way (no imaginary time, though in future one has to investigate the consequences of an imaginary time, too).

ii) For large distances, the old equations of motion of GR,  $\mathcal{R}_{\mu\nu} = 0$ , should arise, i.e., the equivalence to the standard variational principle should emerge. This is because for large  $r$  the Schwarzschild solution should arise.

The condition ii) is automatically fulfilled: Using (67) with (76) and (77) leads for large  $R_-$  to  $\xi_1 \rightarrow (2B)/R_-^4$ . The function  $\xi_0 = \xi_1$  has the same form,  $\xi_3$  is proportional to  $\xi_2$  and  $\xi_2$  also vanishes for large  $R_-$  like  $1/R_-^2$  (see (76)). Thus, for  $R_-$  very large, the  $\xi$  functions tend to zero and the results of standard GR are recovered.

The condition i) is satisfied for

$$g_{00}^0 > 0 \quad . \quad (78)$$

As we will see in the next subsection, the  $M_{\pm}$  values can be set equal to  $m$ . In addition, taking only into account terms up to  $l^0$ , the  $R_{\pm}$  variables are both equal to the radial distance  $r$ . Substituting this into  $g_{00}^0$  component (see (82) below), we obtain a limiting value for  $B$ , solving  $g_{00}^0 = 0$ , i.e.,

$$g_{00}^0 = \left(1 - \frac{2m}{r} + \frac{B}{2r^2}\right) = 0 \quad . \quad (79)$$

We are only allowing real solutions ( $r_0$ ) for the radius variable  $r$ . The solution of the quadratic equation (79) is

$$r_0 = m \left(1 \pm \sqrt{1 - \frac{B}{2m^2}}\right) \quad . \quad (80)$$

When the square root in (80) is different from zero and positive, then there are two real solutions  $r_+$  and  $r_-$ , where the index refers to the sign in (80). Between these two real solutions the  $g_{00}^0$  is negative, which can easily be verified by substituting  $r_0$  into (79). A negative  $g_{00}^0$  would break condition ii) above, thus, it is excluded. In order to avoid a negative  $g_{00}^0$ , the expression in the square root can be at most 0, which implies  $B = 2m^2$  and, thus,  $r_0$  can have at most one real solution. A negative value under the square root implies an imaginary  $r_0$ : For this case, there is no real solution of the quadratic equation (79) and  $g_{00}^0$  is always positive. As we will see in section V, for  $g_{00}^0 = 0$  the redshift will be infinite at half of the Schwarzschild radius, implying a physical division between the interior and exterior of this radius.

We require that all space is connected and, therefore, the limiting solution of  $g_{00}^0$  has to be excluded, too. Nevertheless, we still will discuss this particular case in what follows, interpreting it as a limit of  $B = (2 + \epsilon)m^2$ , with a small  $\epsilon$ . Imposing the above restrictions leads to the condition

$$B > 2m^2 \quad . \quad (81)$$

For a further discussion, we need to enlist the components of the average metric ( $g_{\mu\nu}^0$ ) and, for completeness, the difference metric ( $h_{\mu\nu}$ ). They are given by

$$\begin{aligned} g_{00}^0 &= 1 - \left[ \frac{M_+}{R_+} + \frac{M_-}{R_-} \right] + \frac{\Omega}{2R_-} \\ g_{rr}^0 &= -\frac{1}{2} \left[ \frac{1}{\left(1 - \frac{2M_+}{R_+}\right)} + \frac{1}{\left(1 - \frac{2M_-}{R_-} + \frac{\Omega(R_-)}{R_-}\right)} \right] \\ g_{\theta\theta}^0 &= -\frac{1}{2} (R_+^2 + R_-^2) \\ g_{\phi\phi}^0 &= -\frac{1}{2} (R_+^2 + R_-^2) \sin^2(\theta) \\ h_{00} &= \frac{M_-}{R_-} - \frac{M_+}{R_+} - \frac{\Omega(R_-)}{2R_-} \\ h_{rr} &= -\frac{1}{2} \left[ \frac{1}{\left(1 - \frac{2M_+}{R_+}\right)} - \frac{1}{\left(1 - \frac{2M_-}{R_-} + \frac{\Omega(R_-)}{R_-}\right)} \right] \\ h_{\theta\theta} &= -\frac{1}{2} (R_+^2 - R_-^2) \\ h_{\phi\phi} &= -\frac{1}{2} (R_+^2 - R_-^2) \sin^2(\theta) \quad . \end{aligned} \quad (82)$$

All other elements of the metric are zero.

#### 4.1. Investigating an approximate solution for an exceptional case

For simplicity, we will consider the limiting case of  $B = 2m^2$ , stressing again that one has to add a small value in order that  $g_{00}^0$  is not exactly zero. We analyze some consequences of the solution, obtained above, applying it to the motion of a particle at a distance of  $r \approx m$  from the center, where the  $g_{00}^0$  becomes zero. For simplicity, we approximate the length element  $d\omega^2$  by the expression given in (43), i.e., only the average metric  $g_{\mu\nu}^0$  is considered and only correction up to order  $l^0$  are taken into account. This also implies that  $R_{\pm} \approx r = r \pm l\dot{r}$ . In (43) the variables  $\theta$  and  $\phi$  reduce to real values, with the usual interpretation as azimuthal and polar angles.

With this, the diagonal elements of the average metric, as given in (82), can be approximated by

$$\begin{aligned}
g_{00}^0 &\approx 1 - \frac{M_+ + M_-}{r} + \frac{B}{2r^2} \\
g_{rr}^0 &\approx - \frac{\left(1 - \frac{M_+ + M_-}{r} + \frac{B}{2r^2}\right)}{\left(1 - \frac{2M_+}{r}\right)\left(1 - \frac{2M_-}{r} + \frac{B}{r^2}\right)} \\
g_{\theta\theta}^0 &\approx -r^2 \\
g_{\phi\phi}^0 &\approx -r^2 \sin^2(\theta) \quad .
\end{aligned} \tag{83}$$

Taking into account, that for  $B = 0$  we should get back the standard Schwarzschild metric, suggests the identification of the mass parameters <sup>28</sup>

$$M_+ = M_- = m \quad , \tag{84}$$

with  $m = \frac{GM}{c^2}$ , with  $G$  as the gravitational constant,  $M$  as the mass of the object and  $c$  the light velocity.

We follow closely the steps as indicated in chapter 6.3 of <sup>28</sup>. As shown in Eq. (43) the length element reduces to the usual one ( $d\omega^2 \approx ds^2 = g_{\mu\nu}^0 dx^\mu dx^\nu$ ) with a modified real metric  $g_{\mu\nu}^0$ . As a consequence, the steps to follow will be identical to the standard description of GR. The contributions due to the pseudo-complex structure are simulated by the appearance of the term proportional to  $B \neq 0$ .

The variational procedure ( $X^0 = ct$ ), which has to be applied now, yields

$$\delta \int \left\{ \left(1 - \frac{2m}{r} + \frac{B}{2r^2}\right) c^2 \dot{t}^2 - \frac{\left(1 - \frac{2m}{r} + \frac{B}{2r^2}\right)}{\left(1 - \frac{2m}{r}\right)\left(1 - \frac{2m}{r} + \frac{B}{r^2}\right)} \dot{r}^2 - r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \right\} ds = 0 \quad . \tag{85}$$

The dot indicates now a differentiation with respect to the curve parameter  $s$ .

Varying with respect to the variables  $\theta$ ,  $\phi$  and  $t$ , gives

$$\begin{aligned}
\frac{d}{ds}(r^2 \dot{\theta}) &= r^2 \sin \theta \cos \theta \dot{\phi}^2 \\
\frac{d}{ds}(r^2 \sin^2 \theta \dot{\phi}) &= 0 \\
\frac{d}{ds} \left[ \left(1 - \frac{2m}{r} + \frac{B}{2r^2}\right) \dot{t} \right] &= 0 \quad .
\end{aligned} \tag{86}$$

The second equation gives the conservation of the angular momentum. This implies that the motion is on a plane and one can choose  $\theta = \frac{\pi}{2}$ . With this value, the second equation in (86) yields  $r^2 \dot{\phi} = h = \text{const}$ . The third equations yields

$$\left(1 - \frac{2m}{r} + \frac{B}{2r^2}\right) \dot{t} = \gamma = \text{const} \quad . \tag{87}$$

An additional condition is obtained by dividing the line element  $ds^2$ , as it appears in the integrand of (85), by itself, in complete analogy to the subsection 6.2 of <sup>28</sup>, which gives

$$1 = \left(1 - \frac{2m}{r} + \frac{B}{2r^2}\right) c^2 \dot{t}^2 - \frac{\left(1 - \frac{2m}{r} + \frac{B}{2r^2}\right)}{\left(1 - \frac{2m}{r}\right)\left(1 - \frac{2m}{r} + \frac{B}{r^2}\right)} \dot{r}^2 - r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) . \quad (88)$$

Now, we denote a derivative with respect to the variable  $\phi$  by a prime. For example

$$r' = \frac{dr}{d\phi} = \frac{\dot{r}}{\dot{\phi}} .$$

From this equation and  $r^2 \dot{\phi} = h$ , setting  $\theta = \frac{\pi}{2}$  (motion in a plane), we obtain

$$\dot{r} = \dot{\phi} r' = \frac{h}{r^2} r' .$$

Using (87) and the above deduced relation of  $r^2 \dot{\phi} = h$ , leads to

$$1 = \left(1 - \frac{2m}{r} + \frac{B}{2r^2}\right)^{-1} c^2 \gamma^2 - \frac{\left(1 - \frac{2m}{r} + \frac{B}{2r^2}\right)}{\left(1 - \frac{2m}{r}\right)\left(1 - \frac{2m}{r} + \frac{B}{r^2}\right)} \frac{h^2}{r^4} r'^2 - \frac{h^2}{r^2} .$$

As a next step the variable

$$u = \frac{1}{r}$$

is introduced. The relation of the differential of  $r$  with respect to  $\phi$  to the one of  $u$  is

$$r' = -\frac{u'}{u^2} .$$

Multiplying (89) with  $\frac{1}{h^2} \left(1 - \frac{2m}{r}\right) \left(1 - \frac{2m}{r} + \frac{B}{r^2}\right)$  and using the substitution (89), gives

$$\begin{aligned} & \frac{1}{h^2} (1 - 2mu) (1 - 2mu + Bu^2) = \\ & \frac{(1-2mu)(1-2mu+Bu^2)}{h^2(1-2mu+\frac{B}{2}u^2)} c^2 \gamma^2 - \left(1 - 2mu + \frac{B}{2}u^2\right) u'^2 \\ & - u^2 (1 - 2mu) (1 - 2mu + Bu^2) \end{aligned}$$

22 *Peter O. Hess and Walter Greiner*

Dividing it by  $(1 - 2mu + \frac{B}{2}u^2)$  and solving for  $u'^2$  leads to the equation of motion

$$u'^2 = \frac{c^2\gamma^2}{h^2} \frac{(1 - 2mu)(1 - 2mu + Bu^2)}{(1 - 2mu + \frac{B}{2}u^2)^2} - u^2 \frac{(1 - 2mu)(1 - 2mu + Bu^2)}{(1 - 2mu + \frac{B}{2}u^2)} - \frac{(1 - 2mu)(1 - 2mu + Bu^2)}{(1 - 2mu + \frac{B}{2}u^2)h^2} . \quad (89)$$

Setting  $B = 0$  results in the standard equation of GR for the planetary motion.

This equation is still difficult to solve. We, therefore, recur to a further approximation. In the last section we showed that for  $B = 2m^2$  at  $r = m$  the metric component  $g_{00}^0$  becomes zero. *This is finally the exceptional case we will study, expanding  $u$  around this minimum, i.e.,*

$$u = \frac{1}{m} \left(1 + \frac{\varepsilon}{2}\right) . \quad (90)$$

The factor  $\frac{1}{2}$  in front of  $\varepsilon$  is for convenience.

We obtain for some important factors, appearing in (89), setting  $B = 2m^2$ ,

$$(1 - 2mu) = -(1 + \epsilon) \\ (1 - 2mu + \frac{B}{2}u^2) = \frac{\epsilon^2}{4} . \quad (91)$$

This helps to determine the constant  $\gamma$ , introduced in (87). Inspecting the equation (87) in (86) leads to

$$\frac{\epsilon^2}{4} \dot{t} = \gamma . \quad (92)$$

This has to be fulfilled for any value of  $\epsilon$ , especially when  $\epsilon = 0$ . Therefore, for  $r$  near  $m$  and  $B = 2m^2$  the  $\gamma$  has to be zero! This is valid in particular for orbitals around  $r = m$ .

Taking into account only the leading terms in (89), we arrive at

$$(\epsilon')^2 \approx \frac{16m^2}{\epsilon^2} \left[ \frac{1}{h^2} + \frac{1}{m^2} \right] . \quad (93)$$

Deriving it again, yields

$$2\epsilon'\epsilon'' \approx -\frac{32m^2}{\epsilon^3} \left[ \frac{1}{h^2} + \frac{1}{m^2} \right] \epsilon' . \quad (94)$$

There are two solutions. The first corresponds to  $\epsilon' = 0$ , i.e.  $\epsilon = \text{const.}$  This is the circular motion around the heavy mass object at a distance  $r = m$ . The other one is obtained for  $\epsilon' \neq 0$ . We get

$$\frac{d^2\epsilon}{d\phi^2} \approx -\frac{16m^2}{\epsilon^3} \left[ \frac{1}{h^2} + \frac{1}{m^2} \right] \quad . \quad . \quad (95)$$

Without solving it, we can already deduce some properties from this equation. When  $\epsilon < 0$  ( $r > m$ ) the acceleration  $\epsilon''$  is positive, which translates into  $r''$  negative, i.e., the particle is accelerated toward the center. In contrast, when  $\epsilon > 0$  ( $r < m$ ) the acceleration  $\epsilon''$  is negative, which translates into  $r''$  positive, i.e., the particle is accelerated away from the center.

We emphasize: *At  $r > m$  a particle experiences an attraction toward the center, while at  $r < m$  there is a repulsion!* This implies an **anti-gravitational interaction** for  $r < m$ ! In different words: a massive body continues to contract to lower values of the Schwarzschild radius, until for  $B = 2m^2$  it reaches  $r = m$ . For smaller values the heavy mass object feels repulsion. The body may realize an oscillatory type of motion around  $r = m$ , but this motion is not easy to describe, as can be seen even by the above simplified equation of motion.

The details change when  $B > 2m^2$ , but the cross structure remains.

The contribution of  $B$  seems to be equivalent to the introduction of a  $r$ -dependent cosmological function  $\Lambda(r)$ , though, here it has a different origin and details have still to be worked out. It would be interesting to develop a model for the evolution of the universe, resolving the modified Einstein equations with the contributions of  $B$  and assuming a constant mass distribution in the universe (Robertson-Walker). What will be the possible dependences of  $\Lambda$  as a function in time and what will be its value? This consideration we will leave for a later publication.

## 5. The redshift

In this section we calculate the deviation of the redshift, comparing the present theory with GR. The redshift is an important observable and the detection of possible deviations to known results might be in reach for experiment in near future.

Of particular interest for the redshift is  $g_{00}^0$ . For distances larger than the Schwarzschild radius, the solution is very similar to the standard Schwarzschild solution. However, differences will appear near and below the Schwarzschild radius. *First of all, there is no singularity!* In addition, inside this radius the time component of the metric is positive definite.

Taking all spatial distances to zero ( $dr = d\theta = d\phi = 0$ ), we have to lowest order in  $l$  that  $d\omega^2 \approx d\tau^2 \approx g_{00}^0(r)dt^2$ , with  $\tau$  as the eigen-time. From this we obtain for the change of frequency

$$\nu \approx \sqrt{g_{00}^0(r)} \nu_0 \quad , \quad (96)$$

where  $\nu_0$  is the frequency of a photon at the emission point  $r$  and  $\nu$  is the observed one at large distance.

The redshift  $z$  is defined as <sup>28,32</sup>

$$z = \frac{\nu_0}{\nu} - 1 = \frac{1 - \sqrt{g_{00}^0}}{\sqrt{g_{00}^0}} . \quad (97)$$

In Ref. <sup>33</sup> the  $g$ -factor is defined, which is in the following relation to the redshift:

$$g = \frac{\nu}{\nu_0} = \frac{1}{1 + z} . \quad (98)$$

$g = 1$  corresponds to a flat space ( $z = 0$ ), while  $g < 1$  indicates the relativistic effect ( $z > 0$ ) and  $g = 0$  corresponds to an infinite redshift ( $z = \infty$ ).

Let us discuss the consequences for the solution obtained in the former section: We obtain

$$\nu \approx \sqrt{1 - \frac{2m}{r} + \frac{B}{2r^2}} \nu_0 . \quad (99)$$

The  $g$ -factor is just the square root expression in front of  $\nu_0$ . For the two cases of  $B = 2m^2$  and  $B = 2.2m^2$ , its behavior is depicted in Figs. 1, 2 respectively. Please, note the strong anti-gravitational behavior below half of the Schwarzschild radius. For comparison, the  $g$ -factor for the Schwarzschild solution is depicted in Fig. 3. For  $B = 2.2m^2$  there is no real solution of  $g_{00}^0 = 0$ , i.e.,  $g_{00}^0$  is always positive. Using the value  $B = 2m^2$ , produces a zero at  $x = \frac{r}{2m} = 0.5$ , i.e., an infinite redshift. The value of  $B$  cannot be smaller, otherwise  $g_{00}^0$  would become negative for a certain the range of  $r$ . Larger values of  $B$  will produce  $g$ -factors which are always larger than zero, with no infinite redshift.

For  $r$  toward 0 a blueshift is obtained. The blueshift is an effect of the anti-gravitational force as deduced in the last section.

In Table I, some key values, like the  $g$ -factor and redshift (in parenthesis) are given for several values of  $r$ . From this table we also see, that at the distance  $r = 4m$  from the center, the deviation from the standard Schwarzschild solution is still minimal.

Note, that the redshift is finite and should be measurable near the Schwarzschild radius of giant masses in the center of active galaxies. In <sup>33</sup> a method is presented how to deduce from broad X-ray emission lines the redshift as a function of the radial distance. Results of measured redshifts for the galaxy MrK110 are presented. Unfortunately, the closest distance reported is twice the Schwarzschild radius. Inspecting Table I shows that a notable difference between our calculations and the standard Schwarzschild solution only appears below this distance.

In the extended GR no singularities appear and, thus, black holes in the literary sense do not exist. Very large mass concentrations should appear instead and will be rather gray.



$\frac{r}{2m}$	Schwarzschild	$B = 2m^2$	$B = 2.2m^2$
0.125	- (-)	3.00 (-0.67)	3.26 (-0.69)
0.25	- (-)	1.00 (0.00)	1.18 (-0.15)
0.50	- (-)	0.00 ( $\infty$ )	0.32 (2.13)
0.75	- (-)	0.33 (2.00)	0.39 (1.56)
1.00	0.00 ( $\infty$ )	0.5 (1.)	0.52 (0.92)
1.25	0.45 (1.22)	0.60 (0.67)	0.61 (0.64)
1.50	0.58 (0.72)	0.67 (0.50)	0.67 (0.49)
1.75	0.65 (0.54)	0.71 (0.40)	0.72 (0.39)
2	0.71 (0.41)	0.75 (0.33)	0.75 (0.33)
3	0.82 (0.22)	0.83 (0.20)	0.83 (0.20)
4	0.87 (0.15)	0.88 (0.14)	0.88 (0.14)
5	0.89 (0.12)	0.90 (0.11)	0.90 (0.11)

## 6. Conclusions

We have presented a possible algebraic extension for the theory of General Relativity, which does not contain singularities. Pseudo-complex variables and a modified variational principle were used. We obtained a solution, which depends on the additional parameter  $B$ , whose origin is in the pseudo-complex description. We cannot determine the exact values of  $B$ , because it has to be measured experimentally by detecting deviations of, e.g., the redshift as obtained in our theory with respect to standard GR.

A first finding is the deviation of the redshift compared to standard GR. The calculated redshifts are not infinite any more but approach finite values near the Schwarzschild radius. The differences to the standard Schwarzschild solution are small up to  $r \approx 4m$ . With the present state of technology <sup>33</sup>, however, there is a good chance that the deviations may be observed in near future. A measured deviation from the standard solution implies that large central masses appear as rather gray objects.

As a second important result we obtained an anti-gravitational effect for radii smaller than half of the Schwarzschild radius, assuming the particular values  $B = 2m^2$  and  $B = 2.2m^2$ . For other values of  $B$  the scenario is similar, i.e., for  $r$  smaller than a given distance, anti-gravitation appears. As a consequence, heavy mass objects can not contract to a point at  $r = 0$ . The origin of this effect will be further discussed in a forthcoming paper.

The formulation of the extended GR is done in complete analogy to standard GR, which is advantageous. The difference to standard GR is the use of two, in general distinct, metrics  $g_{\mu\nu}^{\pm}$ .

We showed that the algebraic extension of GR to pseudo-complex variables can be achieved in a consistent manner. The formulation permits non-singular solutions. A unique solution for a spherically symmetric mass distribution was obtained, with

the consequences discussed above. Whether the theory is realized in nature remains to be verified by experiment. Possible signatures in the redshift are proposed.

### Acknowledgement

P.O.H. would like to thank the *Frankfurt Institute for Advanced Studies* (FIAS) at Frankfurt am Main for the hospitality and in particular for the excellent atmosphere during his stay in June 2008. We acknowledge financial support from DGAPA-UNAM and CONACyT. Useful discussions with A. Bounames and K. Nouicer are acknowledged.

### References

1. A. Einstein, *Ann. Math.* **46**, 578 (1945).
2. A. Einstein, *Rev. Mod. Phys.* **20**, 35 (1948).
3. C. Mantz and T. Prokopec, arXiv:gr-qc—0804.0213v1, 2008.
4. A. Crumeyrolle, *Ann. de la Fac. des Sciences de Toulouse*, 4<sup>e</sup> série, **26**, 105 (1962).
5. A. Crumeyrolle, *Riv. Mat. Univ. Parma* (2) **5**, 85 (1964).
6. R.-L. Clerc, *Ann. de L'I.H.P. Section A* **12**, No. 4, 343 (1970).
7. R.-L. Clerc, *Ann. de L'I.H.P. Section A* **17**, No. 3, 227 (1972).
8. E. R. Caianiello, *Nuovo Cim. Lett.* **32**, 65 (1981).
9. H. E. Brandt, *Found. Phys. Lett.* **2**, 39 (1989).
10. H. E. Brandt, *Found. Phys. Lett.* **4**, 523 (1989).
11. H. E. Brandt, *Found. Phys. Lett.* **6**, 245 (1993).
12. R. G. Beil, *Found. Phys.* **33**, 1107 (2003).
13. R. G. Beil, *Int. J. Theor. Phys.* **26**, 189 (1987).
14. R. G. Beil, *Int. J. Theor. Phys.* **28**, 659 (1989).
15. R. G. Beil, *Int. J. Theor. Phys.* **31**, 1025 (1992).
16. J. W. Moffat, *Phys. Rev. D* **19**, 3554 (1979).
17. G. Kunstatter, J. W. Moffat and J. Malzan, *J. Math. Phys.* **24**, 886 (1983).
18. G. Kunstatter and R. Yates, *J. Phys. A* **14**, 847 (1981).
19. M. Born, *Proc. Roy. Soc. A* **165**, 291 (1938).
20. M. Born, *Rev. Mod. Phys.* **21**, 463 (1949).
21. P. F. Kelly and R. B. Mann, *Class. Quantum Grav.* **3**, 705 (1986).
22. P. O. Hess and W. Greiner, *J. Phys. G* **34**, 2091 (2007).
23. P. O. Hess and W. Greiner, *Int. J. Mod. Phys. E* **16**, 1643 (2007).
24. K. Greisen, *Phys. Rev. Lett.* **16**, 748 (1966).
25. G. T. Zatsepin and V. A. Kuzmin, *JETP Lett.* **4**, 78 (1966).
26. F. P. Schuller, Ph.D. thesis, University of Cambridge (2003).
27. F. P. Schuller, M. N. R. Wohlfarth and T. W. Grimm, *Class. Quant. Grav.* **20**, 4269 (2003).
28. R. Adler, M. Bazin and M. Schiffer, *Introduction to General Relativity*, (McGraw Hill, New York, 1975).
29. I. L. Kantor, A. S. Solodovnikov, *Hypercomplex Numbers. An Elementary Introduction to Algebra*, (Springer, Heidelberg, 1989).
30. V. Cruceanu, P. Fortuny and P. M. Gadea, *Rocky Mountain J. of Math.* **26**, 83 (1996).
31. K. Yano, *Differential Geometry on Complex and Almost Complex Spaces*, (Pergamon Press, New York, 1965).
32. J. Plebánski and A. Krasinski, *An Introduction to General Relativity and Cosmology*, (Cambridge, Great Britain, 2006).

33. A. Müller and M. Wold, *Astronomy and Astrophysics* **457**, 485 (2006).

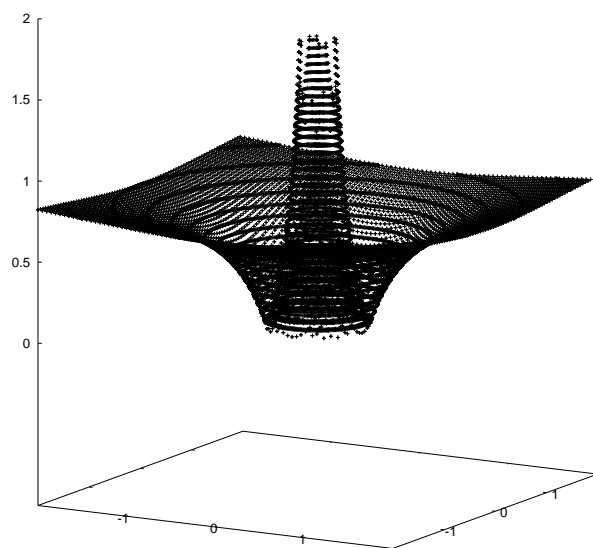


Fig. 1. The  $g$ -factor  $g = \frac{\nu}{\nu_0}$ , for  $B = 2m^2$ , as a function of the coordinates  $x$  and  $y$ , in units of the Schwarzschild radius  $2m$  ( $\frac{r}{2m} = 1$ ), where  $r$  is the radial distance given by  $r = \sqrt{x^2 + y^2}$ . The distance between two contours corresponds to a step size of 0.05.

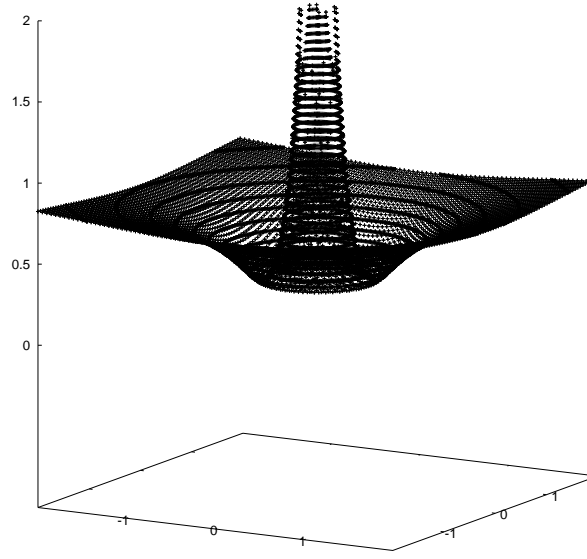


Fig. 2. The  $g$ -factor  $g = \frac{\nu}{\nu_0}$ , for  $B = 2.2m^2$ , as a function of the coordinates  $x$  and  $y$ , in units of the Schwarzschild radius  $2m$  ( $\frac{r}{2m} = 1$ ), where  $r$  is the radial distance given by  $r = \sqrt{x^2 + y^2}$ . The distance between two contours corresponds to a step size of 0.05.

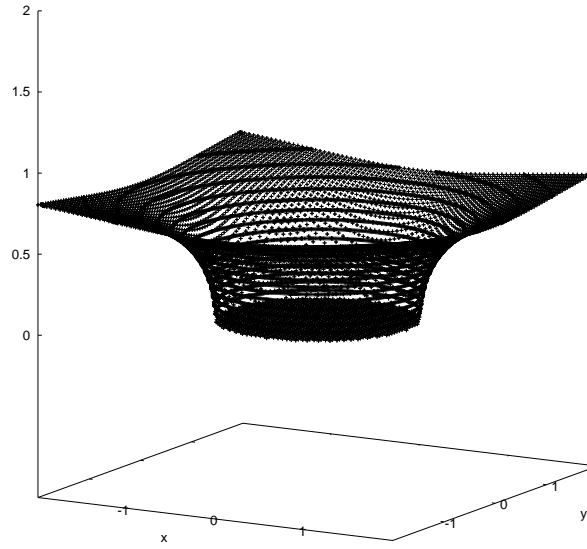


Fig. 3. The  $g$ -factor  $g = \frac{\nu}{\nu_0}$ , for  $B = 0$  the Schwarzschild solution in the standard theory, as a function of the coordinates  $x$  and  $y$ , in units of the Schwarzschild radius  $2m$  ( $\frac{r}{2m} = 1$ ), where  $r$  is the radial distance given by  $r = \sqrt{x^2 + y^2}$ . The distance between two contours corresponds to a step size of 0.05. The Schwarzschild solution is only valid up to  $r = 2m$  where the  $g$ -factor acquires the value 0.